Waveform optimization for phase reconstruction of the impulse response

Christopher W. Tyler and Leonid L. Kontsevich

Smith-Kettlewell Eye Research Institute, 2232 Webster Street, San Francisco, California 94115

Received August 12, 1992; revised manuscript received October 8, 1992; accepted October 13, 1992

The phase spectrum is not available for the responses of systems whose output is integrated over time. A temporal deblurring technique has been described for measurement of the phase spectrum for responses that are based on the integrated output of such systems [Vision Res. **21,** 409 (1981)]. The problem that remains is to determine the optimal sets of Fourier amplitudes to be used for the phase-reconstruction task. Optimal Fourier amplitudes for determination of the phase of each harmonic have now been obtained for relevant weighting functions by an analytic method and by a computation method up to the 20th harmonic. These amplitude sets should permit efficient use of the temporal deblurring paradigm to measure the impulse response of any integrated output system that is linear up to the integration stage.

INTRODUCTION

There is a class of systems for which the output signal is available only through its integrated energy, in which the details of the output waveform are unavailable to analysis. For such systems, which include the perceptual response of the human observer as well as some ultrafast optical phenomena, measurement of the Fourier amplitude spectrum is derived straightforwardly from the amplitude of the integrated response for sinusoidal inputs. The problem lies in the measurement of the Fourier phase spectrum, which is unavailable from the response amplitude. If the phase spectrum can be measured by some indirect procedure, then the impulse response and hence the response waveform can be reconstructed by inverse Fourier transformation.

For such systems the response is given by

$$R(\tau) = \int_{\tau-\Delta}^{\tau} |k(t)|^p \mathrm{d}t, \tag{1}$$

where the inner term is given by the convolution

$$k(t) = f(t) \otimes h(t). \tag{2}$$

The function $h(t)$ represents the system impulse response, and the function $f(t)$ represents the input signal. The constant Δ represents the time over which integration occurs and is assumed to be long relative to the structure of $k(t)$. Thus the output $R(\tau)$ provides a low-pass readout of the integrated response as the integration window Δ passes over the system's response.

The power p is included for generality and is determined by the nature of the system that is performing the integration. If it is a physical energy integrator, then $p = 2$. For psychophysical detection by a human observer,[1–3] p typically falls between 2 and 6. The convolution expression implies that, to the extent that the system is linear at the convolution stage before the integral, knowledge of the impulse response is sufficient to permit characterization of the response [either at the linear stage $k(t)$ or at the output $R(\tau)$] to any arbitrary input.

Since the impulse response $h(t)$ is not directly measurable in such systems, however, it is desirable to be able to reconstruct the response waveform from measurements of the Fourier amplitude and phase characteristics of the system response to sinusoidal input modulations, where

$$h(t) = F^{-1}\{A(\omega)\exp[i\varphi(\omega)]\}, \tag{3}$$

where $F^{-1}\{\}$ denotes the inverse Fourier transform (or its discrete version when restricted to an interval T corresponding to one period of the fundamental frequency of analysis).

Measurement of the response amplitudes, and hence the Fourier amplitude spectrum, is achieved from the integrated response by taking $R^{1/p}$ for sinusoidal inputs. The problem of reconstruction of response before the integration stage lies in the measurement of the Fourier phase spectrum, which is unavailable from R. A sequential phase-deblurring technique for the measurement of the phase spectrum for integrated output systems was recently described.[4] The relative phase can be measured for the (sinusoidal) responses to a complete series of sinusoidal harmonic inputs by the phase-deblurring technique, although the phase of the fundamental response frequency, and hence the overall response delay relative to the stimulus, is indeterminate. The concept underlying this technique is to adjust the phase of each new harmonic to minimize response R over some integration window, Δ, with the previous set of harmonics set to their optimized phases. The phases of the stimulus harmonics then constitute the inverse of the response phase spectrum. The input amplitudes that are required for reaching a threshold response criterion (e.g., $R = 1$) provide the amplitude inverse of the response amplitude spectrum. The temporal impulse response of the system may then be measured by the complex inverse technique of taking the inverse Fourier transform of the inverses of the amplitude and phase spectra for the input stimuli that are required for meeting each criterion.[4]

Two examples of the responses generated during the phase-measurement paradigm are shown in Fig. 1. In

0740-3232/93/051005-09$05.00

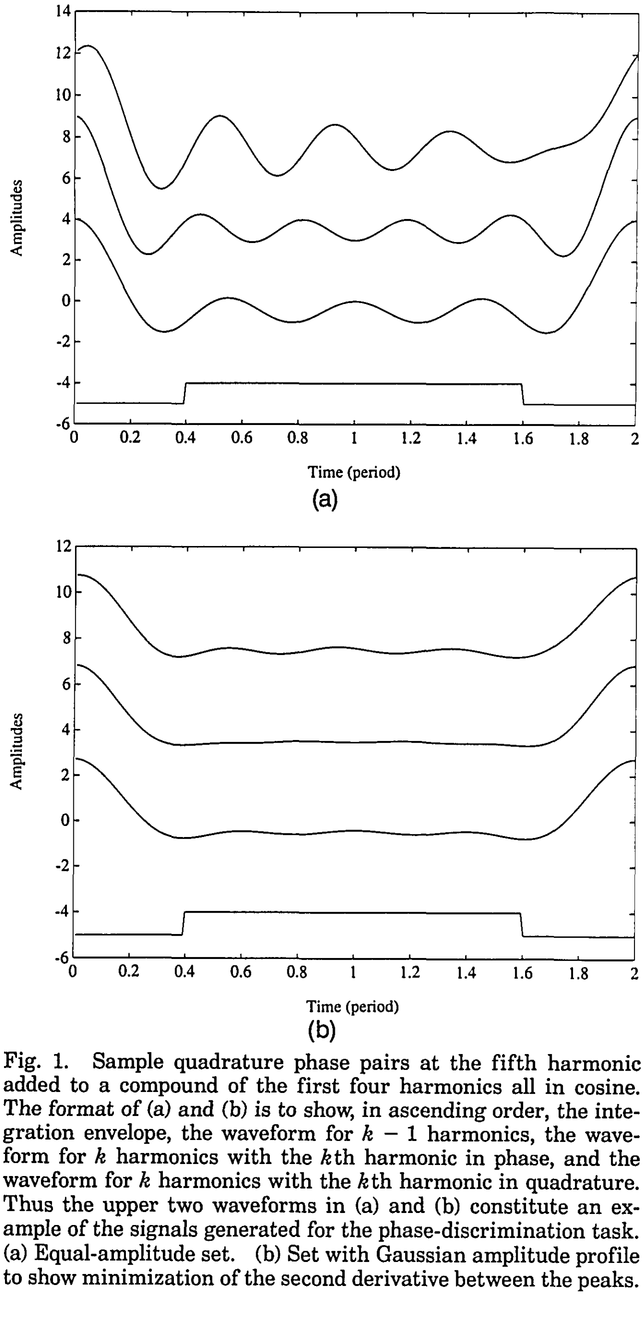

Fig. 1. Sample quadrature phase pairs at the fifth harmonic added to a compound of the first four harmonics all in cosine. The format of (a) and (b) is to show, in ascending order, the integration envelope, the waveform for $k - 1$ harmonics, the waveform for k harmonics with the kth harmonic in phase, and the waveform for k harmonics with the kth harmonic in quadrature. Thus the upper two waveforms in (a) and (b) constitute an example of the signals generated for the phase-discrimination task. (a) Equal-amplitude set. (b) Set with Gaussian amplitude profile to show minimization of the second derivative between the peaks.

Fig. 1(a) the internal response to a waveform consisting of the first four harmonics of a Fourier series is added based on the assumption that the stimulus has been adjusted to equalize the response amplitudes and to put the four harmonics in cosine phase relative to one another (lowest curve). In general, the response waveform is given by

$$k(t) = \sum_{f=1}^{k-1} [a_f \cos(ft) + ib_f \sin(ft)] + [a_k \cos(kt) + ib_k \sin(kt)]. \qquad (4)$$

The summation expression corresponds to the compound waveform of $k - 1$ harmonics (where $k = 5$ for the example in Fig. 1), whereas the remaining terms correspond to the kth harmonic, whose phase is to be determined. For the conditions of this example, all a_f are set to 1 (equal amplitude) and all b_f are set to zero (cosine phase).

The compound waveform [lowest curve in Fig. 1(a)] is the base function for determination of the optimum phase of the fifth harmonic. In the middle curve of Fig. 1(a) the fifth harmonic is added to the compound spike in cosine phase ($a_k = 1, b_k = 0$), again at equal amplitude. In the upper curve the fifth harmonic is added to the compound spike of the lower curve, but now in quadrature (90° phase shift; $a_k = 0, b_k = 1$). The upper pair thus illustrates the internal responses for two key points in the phase-discrimination task. The question addressed in this paper is, What set of response amplitudes is best for performing the phase-optimization task?

Figure 1(b) shows an example of a set of functions that are similar to those in Fig. 1(a), except that the harmonic amplitudes a_f are set according to a Gaussian function that is chosen to optimize the interpeak smoothness when all five harmonics are in cosine phase. These two examples illustrate how, under two quite different amplitude conditions, the fifth harmonic may be largely masked when it is in cosine phase with the previous four harmonics but appears as an extra ripple in the waveform when it is in quadrature. Thus different amplitude sets may produce similar discriminability for the phase variation. Actual measurement of the phase of this latest harmonic may be achieved by a variety of techniques for finding the minimum of its visibility as phase is varied.[4]

RATIONALE

Although the phase-deblurring technique is straightforward in principle, the existence of noise in the system will limit the range of harmonics for which phase measurements are practicable. Noise that is added to the output waveform will limit the variation in integrated response amplitude, and hence the discriminability of the phase for best response, as the phase of the latest harmonic is varied. To maximize this range, therefore, it is important to optimize the phase measurement paradigm to be as robust as possible. This entails setting the amplitude of the phase-varying harmonic at an optimal level relative to the amplitudes of the fixed compound of the previous harmonics against which its phase is being compared. Moreover, the amplitude of each of the previous harmonics that make up the compound may also be set so as to optimize phase discrimination for the lastest harmonic. This adjustment will be different for each new harmonic relative to the first, implying that it is a $k - 1$-dimensional optimization for each set of k harmonics.

To implement amplitude optimization, one must have a model for the criterion involved in the phase-discrimination task. Two such models will be evaluated in this treatment based on the psychophysical optimization procedures reported by practiced human observers.[4] The first model is computational, based on the salience of the second-derivative signal in the compound waveform as relative phase is varied. The second method is an analytic solution for the energy in the waveform weighted by a sinusoidal weighting function at the fundamental frequency of analysis. Since human observers are usually adept at isolating the most relevant cues, these criteria are also expected to be of value in automated procedures for physical signal analysis. The human observers found that the region of the waveform away from the response peak

Fig. 1) permitted the clearest discrimination of the change in perceived waveform structure as phase was varied. The weighting functions in each of the two methods [corresponding to the integration limits in Eq. (1)] were therefore positioned so as to minimize information from the response peak.

The maximization functions to be described were designed to permit optimization of the alignment of the phase of the kth harmonic relative to the compound of the previous $k-1$ harmonics for each harmonic in the series. Once the set of amplitudes that are optimal for maximizing the phase discriminability is determined, the set should enable the phase of the kth harmonic to be aligned at zero phase by empirical minimization of the discriminability function. The maximization functions are developed only for the quadrature phase relation based on the assumption that maximizing the quadrature discriminability will maximize the discriminability to smaller phase variations around zero phase.

It should be obvious that the maximization function must be one that operates in the time domain rather than in the frequency domain because the frequency content of the waveforms remains identical as the phase of the latest harmonic is varied. No criterion that is based solely on amplitude selectivity in the frequency domain would reflect any change in response during phase discrimination, even though the apparent frequency content of the 0° and 90° added-component waveforms in Fig. 1 is quite different. This issue may be clearer if one considers the harmonic series for some waveform with straight segments, such as a sawtooth wave. Such a waveform contains all frequencies, and they are combined in such a way that their presence is not overt. But if one of the harmonics is removed, or its phase is inverted, an explicit ripple at that harmonic frequency appears to be superimposed upon the waveform. The point is that the appearance of such a ripple betrays the deviation from the phase sequence that is required for generation of the straight-segmented waveform and so can be used as a time-domain criterion of the phase mismatch even though the Fourier amplitudes remain invariant.

Second-Derivative Model

The first model for the optimal amplitudes for sequential phase deblurring is based on the concept of maximizing the variation of the second derivative R'' of the waveform as the phase of the latest harmonic is varied. Generally, the second derivative of a waveform is more selective for local variations in a waveform than is the amplitude alone because the former avoids consideration of the dc component and also biases the analysis toward the highest frequency in the waveform. The second derivative therefore seemed to be an appropriate measure to use in the phase-alignment task and also appeared to correspond well to what human observers perceive as the degree of flicker in the temporal waveform.

For the implementation of the second derivative, the actual discriminability measure for the kth harmonic was taken as

$$\hat{R}(f) = \max\left\{\left[\int_{\tau-\Delta}^{\tau} R_{90}''(f,t)^2 \mathrm{d}t\right] \Big/ \left[\int_{\tau-\Delta}^{\tau} R_0''(f,t)^2 \mathrm{d}t\right]\right\}, \tag{5}$$

where

$$R_0(f) = \sum_{f=1}^{k} a_f \cos(ft),$$

$$R_{90}(f) = \sum_{f=1}^{k-1} a_f \cos(ft) + a_k \sin(kt) \tag{6}$$

for temporal frequency f and harmonic amplitudes a_f. The integration limits were set to the values $\Delta = 2\pi - 2T_k$ and $\tau = 2\pi - T_k$ (where T_k is the period of the kth harmonic. This function is shown by the boxcar function at the bottom of each panel of Fig. 3, *infra*). The functions $R_0(f)$ and $R_{90}(f)$ represent the internal responses to the corresponding set of k stimulus harmonics after passage through the system transfer function. The amplitudes and phases of the stimulus that are required for production of these response amplitudes and phases are determined by the temporal deblurring paradigm.[4]

It is necessary to take a rectified form of R'' before integrating because otherwise the value of the integral would tend to zero. This function is therefore analogous to an energy measure over the integrated region Δ. The limits were designed so that the region controlling the maximization of $\hat{R}$ was in the flatter portion of the waveform between the spikes (see Fig. 1). The harmonic amplitudes could be chosen to give this region an almost zero second derivative when all the harmonics were in phase, providing a maximal distinction for the kth harmonic when it was rotated into quadrature. However, the integral limits over the summed waveform make it difficult to derive an analytic expression for $\hat{R}$. The optimal set of amplitudes to maximize $\hat{R}$ therefore was obtained by a computational approach.

Maximization Algorithm

Maximization was implemented by a method of steepest ascent with Boltzmann reinitialization to avoid nonglobal singularities in the maximization function.[5] For a given set of harmonic amplitudes, the steepest slope in their k-dimensional error space was computed. The point of maximal discriminability in that direction was determined, and then a new slope was computed. The sequence was terminated when the largest increment in discriminability corresponded to an increment of $<10^{-6}$ of the total harmonic vector amplitude. Protection against nonglobal singularities in the maximization function was obtained by reinitialization by a random amplitude vector followed by reiteration of the maximization algorithm. Final optimization was assumed to have been achieved if reinitialization did not increase the maximization function after 50 attempts.

Results

The family of optimized amplitudes for the second-derivative criterion of Eq. (1) is provided in Table 1. For every set of k harmonics the amplitudes that maximize phase discriminability are a monotonically decreasing sequence. A sample of the results is depicted in Fig. 2 for $k = 2, 5, 10, 15, 20$ to provide a graphic overview of the results. The optimized amplitude function is not recognizable as a standard analytic function since it has a steeper drop from the first to the second harmonic followed by a slight shoulder and a flattening toward the last harmonic.

Table 1. Optimized Amplitudes for the Second-Derivative Criterion of Eq. (1)

Set of k Harmonics	Amplitudes That Maximize Phase Discriminability																			
1	1	0.2447	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
2	1	0.5019	0.1268	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
3	1	0.6864	0.3392	0.1043	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
4	1	0.794	0.5123	0.2617	0.0949	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
5	1	0.7509	0.5647	0.3615	0.1935	0.0825	--	--	--	--	--	--	--	--	--	--	--	--	--	--
6	1	0.7704	0.6271	0.4548	0.2947	0.1649	0.079	--	--	--	--	--	--	--	--	--	--	--	--	--
7	1	0.7935	0.7078	0.5593	0.4078	0.2709	0.1597	0.0865	--	--	--	--	--	--	--	--	--	--	--	--
8	1	0.808	0.7302	0.6093	0.4769	0.3494	0.2365	0.1146	0.0859	--	--	--	--	--	--	--	--	--	--	--
9	1	0.8169	0.7429	0.6591	0.5433	0.4237	0.3121	0.2145	0.1352	0.0863	--	--	--	--	--	--	--	--	--	--
10	1	0.8346	0.7692	0.6853	0.5823	0.4761	0.3725	0.277	0.1939	0.1258	0.086	--	--	--	--	--	--	--	--	--
11	1	0.8445	0.7788	0.7126	0.6253	0.5291	0.4327	0.3403	0.2559	0.1825	0.1217	0.0901	--	--	--	--	--	--	--	--
12	1	0.8554	0.793	0.7343	0.6584	0.5715	0.4827	0.3954	0.3129	0.238	0.1729	0.1186	0.0935	--	--	--	--	--	--	--
13	1	0.8685	0.8071	0.7538	0.6956	0.6174	0.5341	0.4507	0.3697	0.2939	0.2254	0.1656	0.1155	0.0951	--	--	--	--	--	--
14	1	0.8726	0.8143	0.7644	0.7082	0.6377	0.5613	0.4836	0.407	0.3337	0.2657	0.2046	0.1515	0.1067	0.0906	--	--	--	--	--
15	1	0.8808	0.8229	0.7709	0.7193	0.6587	0.5911	0.5212	0.4511	0.3825	0.3169	0.256	0.2008	0.1523	0.1108	0.1011	--	--	--	--
16	1	0.8867	0.831	0.7799	0.7336	0.6852	0.6268	0.5595	0.4903	0.4224	0.3567	0.2947	0.2376	0.1861	0.1412	0.1031	0.0956	--	--	--
17	1	0.8937	0.8413	0.7937	0.7486	0.6977	0.6396	0.5783	0.5158	0.453	0.3914	0.3322	0.2764	0.225	0.1786	0.1377	0.1026	0.1001	--	--
18	1	0.8972	0.8461	0.8001	0.7606	0.7206	0.6691	0.6151	0.5568	0.4976	0.4388	0.3813	0.326	0.2739	0.2257	0.1819	0.1429	0.1089	0.114	--
19	1	0.9046	0.8567	0.8108	0.7685	0.7264	0.6795	0.6277	0.5738	0.5191	0.4642	0.4101	0.3574	0.3071	0.2597	0.2158	0.1759	0.1401	0.1088	0.1206

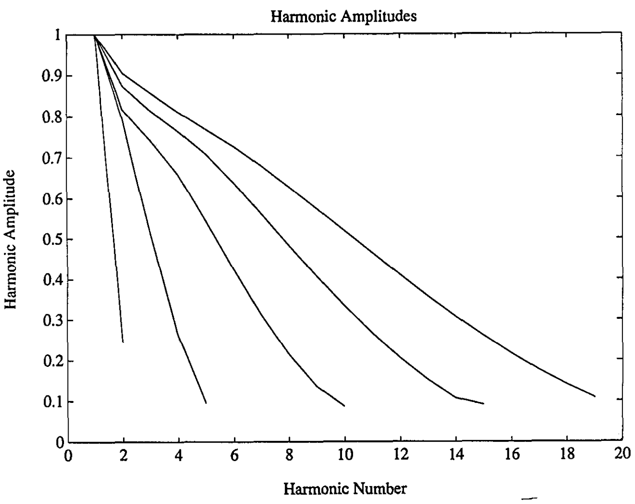

Fig. 2. Optimized amplitudes for sets of harmonics with $k = 2$, 5, 10, 15, 20 for the optimized second-derivative criterion.

A remarkable feature of the optimization results is that the amplitude of the final harmonic falls close to a constant value of ~0.1 for all harmonics. To help to explain the utility of this result, in Fig. 3 we show sample waveforms for the optimized amplitudes for the amplitude sets depicted in Fig. 2. Note that when the phase of the kth harmonic is zero the optimization sets the amplitudes of the harmonics to minimize the waveform fluctuations in the defined minimization region. When the kth harmonic is switched to quadrature phase, it is then revealed in almost its full amplitude. The fact that the optimized amplitude for the latest harmonic is almost constant would imply that the phase discriminability of the kth harmonic itself would be similarly invariant in the absence of measurement noise.

Measurement Noise

Measurement noise is, of course, unavoidable in any empirical technique. The noise will be cumulative and may be considered to be an added vector of $k - 1$ harmonics of random amplitude. If the harmonic phases are measured in sequence starting with the lowest one,[4] then the noise against which each new harmonic is measured is larger by the vector sum of the noise from the previous harmonics. This gives rise to a series describing the variance of the noise at each harmonic defined by

$$\begin{aligned} V_k &= \nu_2 + (\nu_2 + \nu_3) + (\nu_2 + \nu_3 + \nu_4) + \ldots \\ &= (k-1)\nu_2 + (k-2)\nu_3 + \ldots + \nu_k . \end{aligned} \tag{7}$$

If we assume that the noise has an equal effect on the minimization task for all harmonics, it can be shown that V_k is an exponential series such that

$$V_k = \exp(bk), \tag{8}$$

where $b \approx 0.8$.

However, the use of a criterion such as the second derivative substantially reduces the effect of the noise from

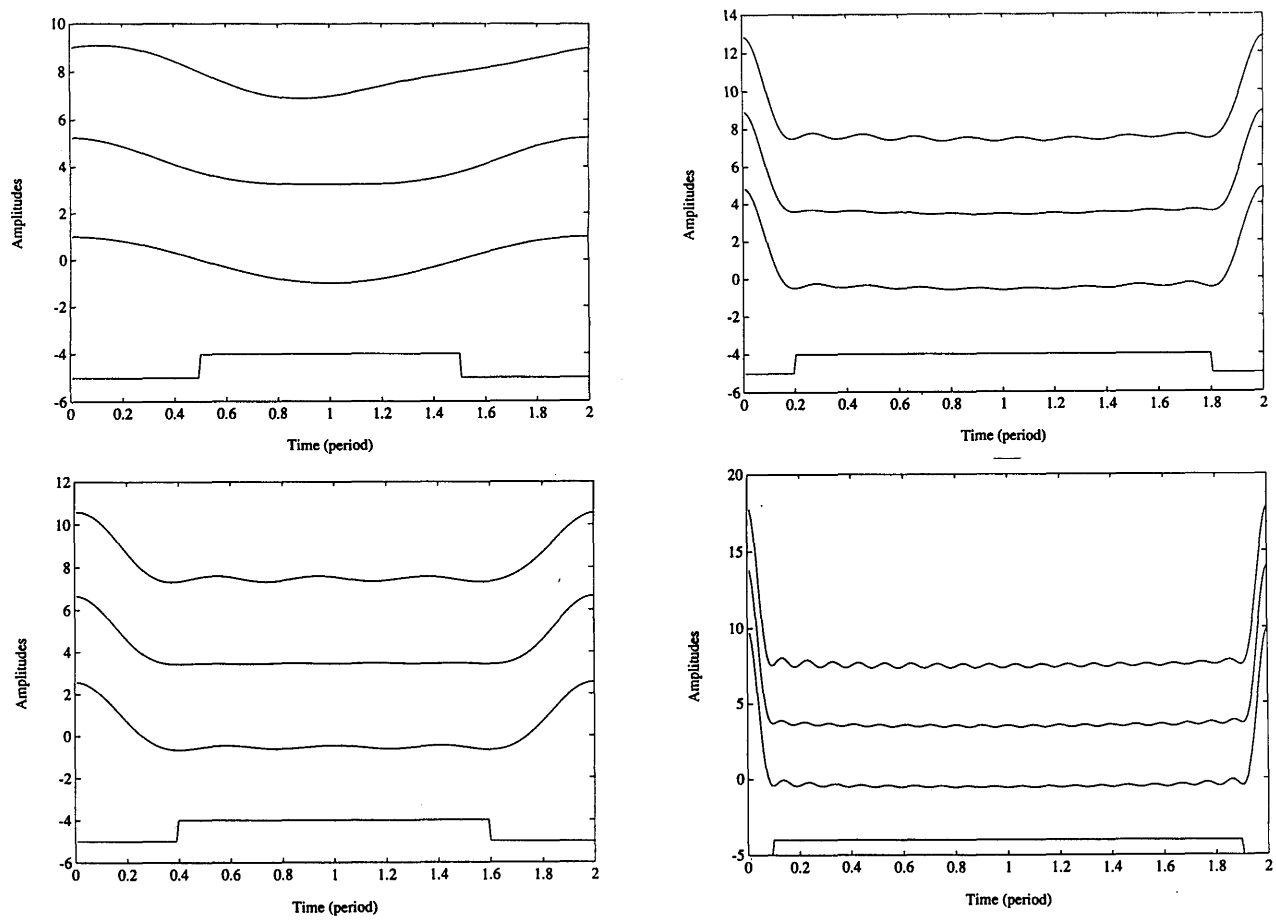

Fig. 3. Quadrature waveform pairs in the format of Fig. 1 for the optimized amplitude sets of Fig. 2. The boxcar at the foot of each panel shows the region of integration for the response analysis.

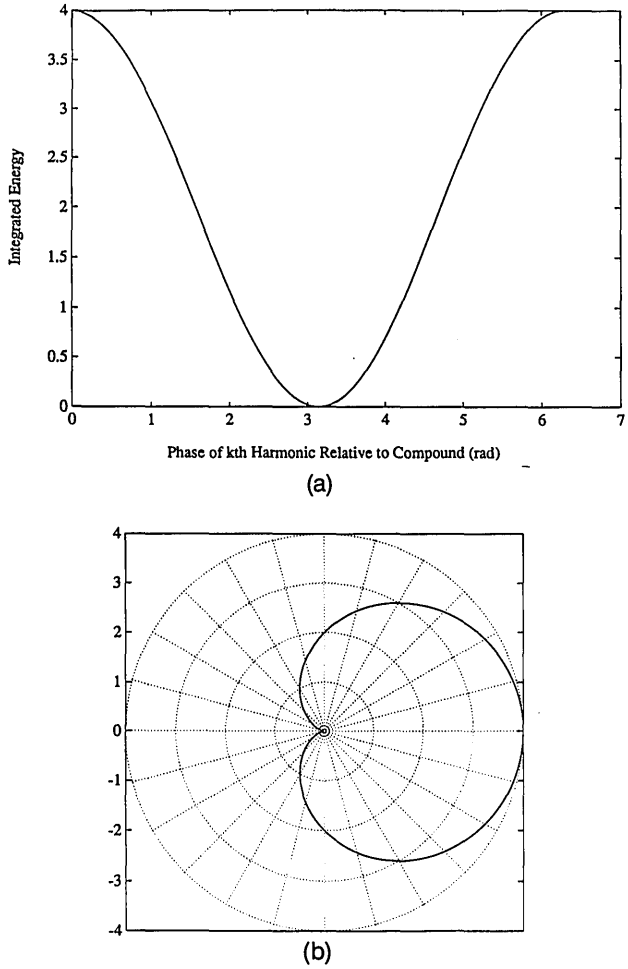

Fig. 4. Approximate energy-phase function for the addition of the kth harmonic to the compound of $k - 1$ harmonics. Energy integrated over the half-cycle farthest from the response maximum. (a) Linear coordinates. (b) Polar plot, where the ordinate specifies the vertical meridian.

the harmonics below the current one in accordance with the square of the frequency of each harmonic, so the exponential series would not be so steep as is implied by this upper bound.

It is worth noting that the noise levels that are obtained from the sequential technique are not the limiting values for phase measurement. Suppose that we pursue a series up to the highest harmonic that is measurable by a sequential technique. At this point the waveform is a noisy version of the type shown in Fig. 3. Having reached the highest measurable harmonic, we may now go back and optimize first the amplitude and then the phase of each harmonic in descending sequences, with the full complement of harmonics as a background. This should permit a notable reduction in the noise from that obtained from the original ascending sequence.

Application to Phase Determination

After one derives the optimum amplitude set, how can one implement it in a phase-discrimination task? For physical measurement of the response energy, it should be possible to limit the energy measurement to the half-cycle opposite the response peak. In this region of the waveform the energy of the compound of $k - 1$ harmonics may be considered to constitute a base vector whose phase is unknown. The kth harmonic then corresponds to a vector added in various phases to generate an energy function corresponding to the combined amplitude of the two vectors. The variation of this energy function with the phase of the kth harmonic is shown in Fig. 4 in both linear and polar formats.

For human psychophysics the best methodology is a forced-choice discrimination task, in which the observer must choose between two different stimulus configurations on each of a large number of trials. This kind of task can be adapted to determine the phase of each harmonic by measuring discrimination performance between a pair of phases for the kth harmonic as the pair is rotated with respect to the compound of $k - 1$ harmonics. For example, a series of stimuli is presented with the kth harmonic randomly in either phase φ or phase $\varphi - 90°$. The observer must discriminate which phase was present on each trial, presumably on the basis of which stimulus looked the least wiggly (had the lowest second derivative). This discrimination will be optimal, i.e., it will show the highest percent-correct responses, when the phase of either component is at zero relative to the phases of the harmonics that make up the compound.

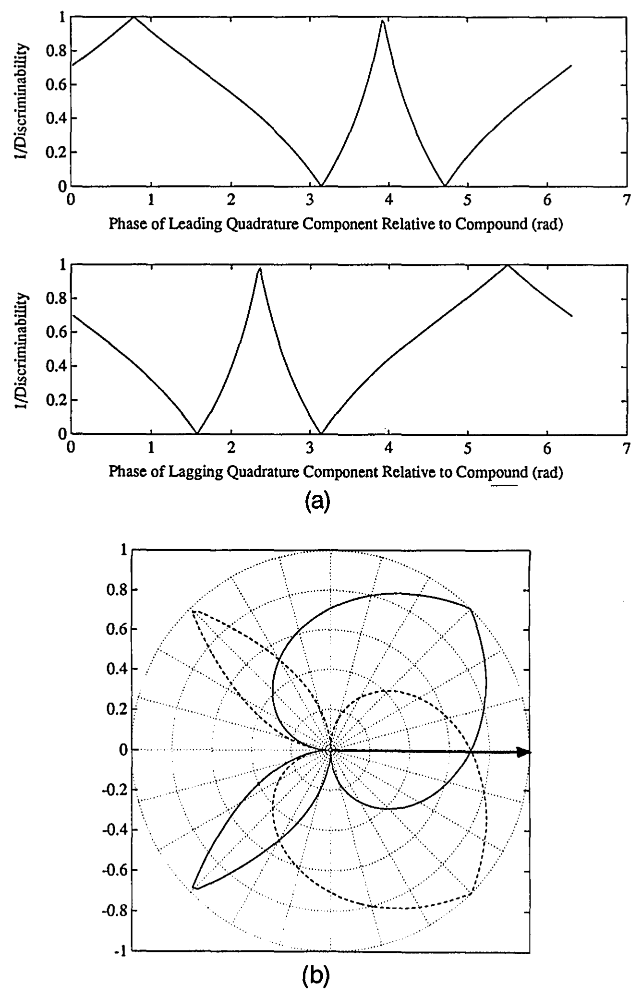

Fig. 5. Discriminability between a pair of harmonics added in quadrature as a function of the absolute phase of each member of the pair. (a) Reciprocal discriminability as a linear function of the phase of each added harmonic. (b) Polar plot of both harmonics to show phase determination rule.

Measurement of the discriminability as the phase of the first component is varied will generate a discrimination function of the form shown in Fig. 5(a), where the ordinate is expressed as the reciprocal of discriminability. The two zeros correspond to the points when the first component and its quadrature companion are in the optimum phase. The broad and narrow peaks correspond the points when the two harmonics are at ±45° and ±135°, respectively, in relation to the optimum. The asymmetry of this discrimination waveform permits unambiguous determination of the point of optimum phase of the kth harmonic, for all harmonics except the second. This determination may be easily performed by plotting the discriminability in polar coordinates, as shown in Fig. 5(b). The two club-shaped forms point 45° to either side of the optimum phase for the kth harmonic (arrow).

For the second harmonic the waveform has two maxima with no asymmetry in discriminability to distinguish them. In principle, the situation may be disambiguated either by making an arbitrary choice or by asking the observer to make the additional judgment as to which of the points corresponded to a bright narrow peak and which corresponded to a dark narrow peak. However, in practice there is a perceptual asymmetry that can disambiguate the two alternatives. It appears that human detection thresholds for larger targets exhibit a threshold asymmetry in which the narrow dark peak is more readily detectable than the narrow bright peak.[4,6–8] This should serve adequately to disambiguate the second harmonic case on the basis of discriminability alone.

Analytic Solution for a Particular Case

The problem of maximization of the phase discriminability of the latest harmonic can be solved analytically on the basis of a tractable weighting of the energy for different parts of the periodic signal. Consider the weighting function

$$w(t) = 1 - \cos(t)$$

for the periodic signal $S(t)$ compound of the first N Fourier harmonics. Suppose that the system analyzing this signal estimates the energy

$$E = \int_0^{2\pi} [w(t)S(t) - \overline{w(t)S(t)}]^2 \mathrm{d}t, \tag{9}$$

where

$$\overline{w(t)S(t)} = \int_0^{2\pi} [w(t)S(t)\mathrm{d}t/2\pi] \tag{10}$$

for all harmonics (excluding the dc component) of the weighted signal simultaneously. For the given $w(t)$ it means that the analyzing system is insensitive to variations of the signal at the phases close to $t = 2\pi m$ and sensitive to them near $(2m + 1)\pi$ for all integers m.

As long as the task is to set the kth harmonic in a definite phase based on the energy information, this phase must be a special point in the functional relation between the kth harmonic phase and energy of the weighted signal. The main candidates for this point are the extremum points. Here we shall consider the minimum point, which corresponds to the task for the analyzing system to set the phase of the kth harmonic such that the signal would be as small as possible when the analyzing system is most sensitive.

Thus the task is to find the signal that provides the maximum discriminability at the minimum weighted energy. We shall consider two natural definitions for this discriminability. The first is based on the comparison of energy for quadrature of the kth harmonic:

$$\frac{E_{\varphi_{\min}} + 90°}{E_{\varphi_{\min}}}, \tag{11}$$

where E_φ is the energy for the kth harmonic with phase equal to φ and $\varphi_{\min}$ is the phase where energy reaches its minimum. Another definition is

$$E_{\varphi_{\min}}''/E_{\varphi_{\min}}, \tag{12}$$

which may be considered a version of the Weber fraction; many human perception mechanisms discriminate signals on its basis. We shall show that, for the given weighting function, the two definitions of the discriminability lead to almost indistinguishable results.

We start from two propositions:

Proposition 1. If the energy function can be decomposed to the sum of two nonnegative components of which one depends and the other does not depend on the kth harmonic, then, for both definitions of discriminability, the discriminability improves when the second component decreases.

Proof. The statement directly follows from the obvious inequality

$$(a + b)/(a + c) < b/c \qquad \text{for any positive } a, b, c, \text{ while } b > c. \tag{13}$$

Proposition 2. The complex Fourier spectrum of the $S(t)$ that minimizes both criteria must be a linear function of the frequency in the segment $(0, N)$.

Proof. Consider a Fourier spectrum of the $w(t)S(t)$:

$$F[w(t)S(t)] = w(f) \otimes S(f), \tag{14}$$

where $\otimes$ denotes convolution. In the frequency domain, the weighting function values are

$$w(-1) = w(1) = -1/2, w(0) = 1 \tag{15}$$

and zero at all other points. Therefore the convolution with $w(f)$ is equivalent to calculation of the discrete second derivative, which means that Fourier components of the optimal signal $S(t)$ on the segment $(0, N - 1)$ compose a straight line (because energies of the harmonics below N reach the minimal zero value). Thus, if $w(N - 1)$ and $w(N)$ are given, then the spectrum of the optimal signal must belong to the line that passes through these two points. The proposed linearization of the lower part of the signal spectrum can lead to inconsistency where $\mathrm{Im}[w(0)] \neq 0$. This inconsistency can be avoided by turning all harmonics by the angle required to make the dc component real. This global turn can change the value of $\varphi_{\min}$ but does not affect internal phase relations (and, consequently, discriminability at $\varphi_{\min}$) of the signal.

Now we can estimate E_φ for the signal with the linear spectrum. As long as the Fourier spectrum of the signal can be known up to the multiplier, let the kth harmonic have unit amplitude, or, in other words, $S(k) = \exp(i\varphi)$.

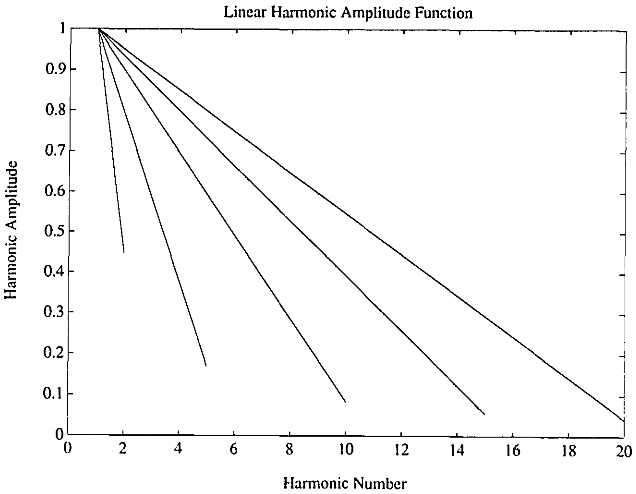

Fig. 6. Optimized sets of harmonics with $k = 2, 5, 10, 15$, and 20 for the analytic energy criterion with sinusoidal weighting function.

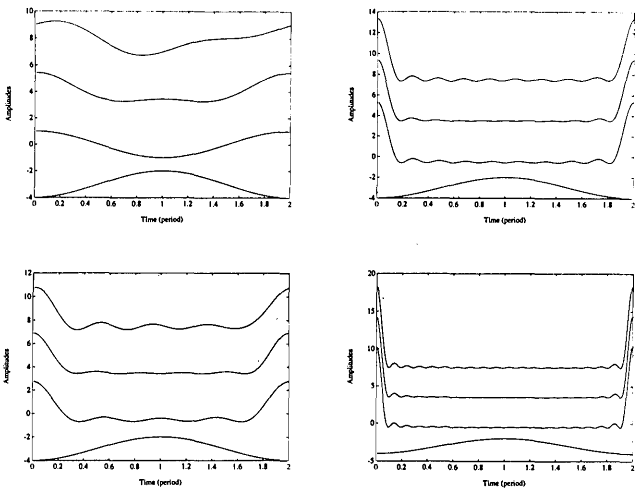

Fig. 7. Sample waveforms for harmonic sets in the format of Fig. 1 under the analytic energy criterion of Fig. 6. The sinusoid at the foot of each panel shows the weighting function for the response analysis.

Consider the kth and $k + 1$st harmonics of the convolution (only these components cannot be nullified by the linearization of the spectrum):

$$F[w(t)S(t)](k) = \exp(i\varphi) - S(k - 1)/2, \tag{16}$$

$$F[w(t)S(t)](k + 1) = \exp(i\varphi)/2, \tag{17}$$

Consequently,

$$E_\varphi = S(k)^2 + 5 - 2S(k)[\exp(i\varphi) + \exp(-i\varphi)]. \tag{18}$$

Remember that $\varphi_{\min}$ must be at the extremum of energy. Therefore

$$\frac{\partial E}{\partial \varphi} = -2iS(k)[\exp(i\varphi) - \exp(-i\varphi)] = 0,$$

and hence

$$\varphi_{\min} = 0, \tag{19}$$

which means that the optimal signal for both discriminability criteria has a linear real spectrum.

The only remaining free parameter is the slope, which may be found for both discriminability criteria. Consider the first definition.

Let

$$x = S(k - 1). \tag{20}$$

Then

$$\frac{E_{90}}{E_0} = \frac{x^2 + 5}{x^2 - 4x + 5}. \tag{21}$$

The minimum of this ratio that provides the lesser slope of the amplitude spectrum is $x = \sqrt{5}$. Thus the amplitude of each harmonic h of the optimal signal is

$$S(h) = S(-h) = k - h)\sqrt{5} + h - k + 1 \qquad \text{for} \quad h \leq k, \tag{22}$$

$$S(h) = S(-h) = 0 \qquad \text{for} \quad h > k. \tag{23}$$

For the second definition, x must fit the condition

$$\frac{\partial}{\partial x} \frac{\dfrac{\partial^2 E_0}{\partial \varphi^2}}{E_0} = 0, \tag{24}$$

which leads to the solution

$$x = (\sqrt{29} - 1)/2, \tag{25}$$

which is quite close to that of Eq. (21).

The derived weighting functions are shown in Fig. 6 for comparison with the optimization results in Fig. 2. They evidently provide an intuitively valid solution, as illustrated in Fig. 7, which shows the corresponding response waveforms for different numbers of harmonics in the format of Fig. 3. Although the functions are superficially similar to those for the optimization algorithm in Fig. 2, they are less advantageous in practice because the amplitude of the kth harmonic decreases more rapidly for the mathematical derivation than for the optimization. By the 20th harmonic, the amplitude of the final harmonic is only 4% of the fundamental, as compared with 12% for the optimization, although the amplitudes are similar when $k = 10$. The difference is therefore relevant only when a large number of harmonics is required in the phase reconstruction.

CONCLUSION

Although there is no general solution of the amplitudes required for phase reconstruction in the temporal deblurring paradigm, optimal Fourier amplitudes for determination of the phase of each harmonic have now been obtained for both an analytic method and a computation method up to the 20th harmonic. These amplitude sets are quite similar and suggest that the solution to this optimization task is relatively robust to the assumed form of the integration window. The optimization results should permit efficient use of the temporal deblurring paradigm to measure the impulse response of any integrated output system that is linear up to the point of integration.

ACKNOWLEDGMENT

This research was supported by National Science Foundation grant BNS 90011837.

REFERENCES

1. J. G. Robson and N. Graham, "Probability summation and regional variation in contrast sensitivity across the visual field," Vision Res. **21,** 409–418 (1981).
2. M. J. Mayer and C. W. Tyler, "Invariance of the slope of the psychometric function with spatial summation," J. Opt. Soc. Am. A **3,** 1166–1172 (1986).
3. H. R. Wilson and J. Bergen, "A four mechanism model for threshold spatial vision," Vision Res. **19,** 19–32 (1979).
4. C. W. Tyler, "Psychophysical derivation of the impulse response through generation of ultrabrief responses: complex inverse estimation without minimum-phase assumptions," J. Opt. Soc. Am. A **9,** 1025–1040 (1992).
5. J. Hertz, A. Krogh, and R. G. Palmer, *Introduction to the Theory of Neural Computation* (Addison-Wesley, Redwood City, Calif., 1992).
6. J. Krauskopf, "Discrimination and detection of changes in luminance," Vision Res. **20,** 671–677 (1980).
7. R. M. Boynton, M. Ikeda, and W. S. Stiles, "Interactions among chromatic mechanisms as inferred from positive and negative increment thresholds," Vision Res. **4,** 87–117 (1964).
8. C. W. Tyler, H. Chan, and L. Liu, "Different spatial tunings for ON and OFF pathway stimulation," Ophthal. Physiol. Opt. **12,** 233–240 (1992).