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# The Vault of Perception: Are Straight Lines Seen as Curved?

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**Abstract**

There is a widespread belief in art praxis that linear perspective is only a geometric approximation to the 'true' properties of perspective as experienced in the perception of the world, which are thought to involve some form of curvilinear perspective. The origins of that belief are examined from Roman times to the present, with a focus on the generation of perspective curvature by the active viewer as a means of elucidating the underlying perceptual principles involved. It is concluded that the only valid form of perspective for the flat canvas is linear perspective, and that it is valid only for a viewing location at the geometric center of projection for which the picture was constructed. Viewing from any other location (particularly in the case of wide-field images viewed from greater than the required distance) generates perceived distortions that have often been misinterpreted to imply that linear perspective geometry is inadequate and that some form of curvilinear perspective would be more representative. However, as long as it is viewed with one eye from the center of projection, the perceptual experience of accurate linear perspective is of a full, explorable 3D space, in contrast to any other form of perspective convention.

**Keywords**

Perspective, linear, curvilinear, history, geometry

## 1. Introduction

The question of the ambiguities in representing the three-dimensional world on a flat plane has a long and illustrious history, with the technique of linear (or geometric) perspective encompassing figures from the Renaissance through to the twenty-first century. What is not often recognized in Renaissance-oriented accounts of the history of perspective, however, is how clear the Roman architect Marcus Vitruvius Pollio was about the principles of perspective and

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the viewpoint of the observer, and how much influence those ideas seem to have had in the development of the perspective schema of the Renaissance (see Note 1). Writing around the time of the establishment of the Roman Empire (~30 BC), Vitruvius describes the convergence of perspective lines to a central vanishing point, and the diminution in the perceived dimensions of objects with distance from the eye (as instantiated in some wall paintings from the era; Little, 1971).

It is worth considering the specific quotation in which Vitruvius describes the perspective construction (under the rubric "Taxis"), which outlines the very same scheme for architectural rendering that has been commonly used for architectural plans up to this day (or at least until the advent of computer renderings):

> "*Taxis* (Organization). The detailed layout, [literally, disposition of the elements] which the Greeks call the concept, is this: *groundplan*, *elevation*, *perspective*. The groundplan is made by the proper sequential use of ruler and compass, through which we map the locations for the plane surfaces of the building. The elevation is a picture of the front of a building, set upright and properly drawn in the proportions of the intended work. The perspective is the method of depicting the front with the sides receding into the background, the lines all meeting at the convergence point. These [elements] derive from analysis and creativity." (Vitruvius I. 2. 2. Author translation and emphasis)

Vitruvius' 'Ten Books on Architecture' was widely available during the classical Renaissance, and most of the well-known figures of the history of perspective were known to have owned a copy (see Note 2). Thus, the similar ideas on perspective that can be found in the works of Filippo Brunelleschi (Manetti, 1480/1927), Leon Battista Alberti (1435), Piero della Francesca, and Leonardo da Vinci are likely to have been derived directly from the cited passage in Vitruvius (see Note 3).

In particular, Leonardo da Vinci distinguished between simple (natural) and complex (artificial) perspective, the natural perspective being that perceived by the observer in the world, and the artificial perspective being that used in paintings. The key distinction is that artificial perspective is how the image of the world is projected onto the flat plane of a painting from the spatial location of one eye of the human viewer, whereas natural perspective refers to the perception of the world itself by the observer, without an intervening plane of projection. The geometry of natural perspective was conceptualized in terms of the sphere of vision by Vitruvius, and its contrast with the projection of a plane image, as depicted in Fig. 1 by a 17th century diagram from Bosse (1665). Natural perspective is much more difficult to specify than artificial perspective, since there is no external image to provide an explicit criterion

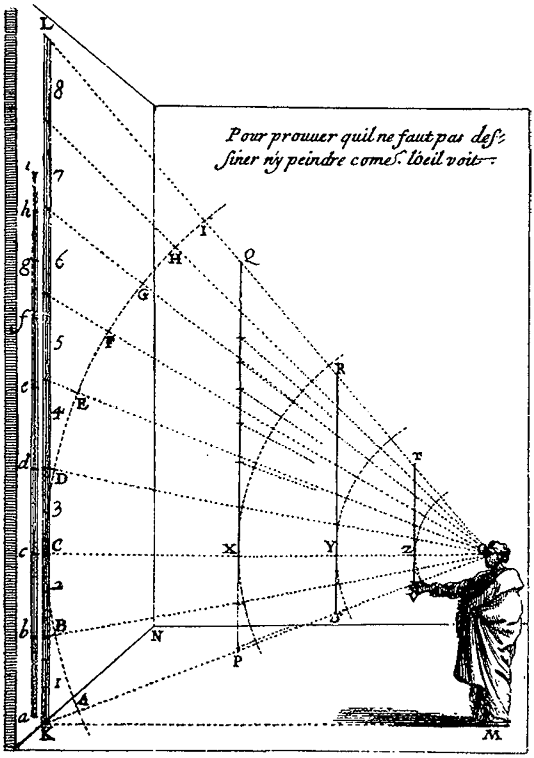


**Figure 1.** Illustration from *Traité des Pratiques Géometrales et Perspectives* by Abraham Bosse (1665). The French epigram can be translated: “To prove that one can neither define nor paint as the eye sees.”

for any construction geometry, and the ultimate criterion for veridicality is the viewer’s *subjective* judgment as to how things appear, which itself depends on the implicit viewing conditions for the observations. In particular, the only way to ascertain whether lines appear straight under natural observation conditions is to ask viewers of physical scenes containing extended straight lines whether the lines *appear* straight or curved.

## 2. Ernst Gombrich and Curvilinear Perspective

Among his many forays into the subtleties of perceptual interpretation, Ernst Gombrich (1972) took on the issue of his forebear, Erwin Panofsky, as to whether linear perspective is in some sense perceived as curved (Panofsky, 1927/1991). In general, Gombrich adhered to the position that the lines of geometric perspective appear straight. Referring to Brunelleschi’s ‘peephole’ depictions of the Florence Baptistery, he said:

> "... I remain an unrepentant straightliner, but at the risk of monotony I should like to restate my case in a form that may perhaps further the argument a little. [...] If he [Brunelleschi] chose to draw this image on a flat panel opposite the peephole, the laws of geometrical projection would come into play which state that planes parallel to the picture plane — for instance the central facade of the baptistery — would appear in the projection as reduced in size but similar in shape — parallels remaining parallels." Gombrich (1972, pp. 139–140)

Despite his self-avowal as an 'unrepentant straightliner', however, it seems that Gombrich was not hostile to the idea of curvature under conditions of extreme angle of view. In 'The Vault of Perception', he says:

> "While observing the vapour trails of jet planes which so frequently disturb the calm of a cloudless sky I have come to appreciate the reasons why some students of art, including the great Panofsky, asserted with such conviction that we 'really' see straight lines as curved... Even though I have reason to think that the course of the jets is straight (see Note 4), I tend to see the trail rising over the horizon and arch around me, flattening overhead, but turning down steeply before it vanishes." (Gombrich, 1974, p. 90)

Why the apparent contradiction with his earlier statement? What seems to have happened here is that, without stating so explicitly, Gombrich has switched from the analysis of pictorial perspective to that of the mobile observer viewing the three-dimensional world. Gombrich is a 'straightliner' in the context of projection to a picture plane. This characterization of straight lines in the world as projecting to straight lines in the picture plane is geometrically correct, as was amply demonstrated by Maurice Pirenne (1970) with his pinhole camera photographs (Fig. 2A). Almost all commentators agree on this point, that the perspective projection of any straight line in the world is a straight line (or, in the limit, a point) in the picture plane to which it is being projected (Tyler, 1998).

It may be relevant to point out that a range of non-Euclidean spatial metrics was adduced by Koenderink, van Doorn and Lappin (2000), which varied as a function of distance from elliptic to hyperbolic to parabolic in their open field (binocular) measurements of triangular pointing directions. A later study of the perceived frontoparallel from the same group (Koenderink *et al.*, 2002) gave the paradoxical result that large-scale frontoparallel planes appeared concave to the viewer. However, it should be emphasized that these studies were performed under literally open-field conditions, with binocular viewing and body rotation requiring integration of the spatial estimates across multiple views with no explicit perspective information (except texture gradient cues). Thus,

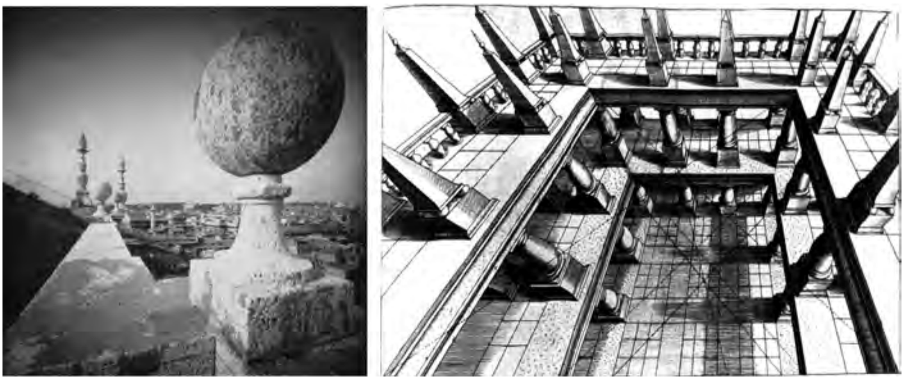

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**Figure 2.** (A) 'Pinhole camera view of the roof of S. Ignazio, Rome' by Maurice Pirenne (1955). (B) 'Courtyard' by Jan Vredeman de Vries (1605). Both images have strong marginal distortions away from the center of the picture that are apparent under the usual viewing conditions. However, the perspective in both cases is geometrically correct for a viewing distance of about half the picture width. To view them comfortably under these conditions, each picture needs to be expanded to about 1 m in width, when the marginal distortions will be perceived to disappear and the pictures will be perceived in vivid 'stereoscopic' depth.

they are not directly relevant to the issue of pictorial space with perspective information.

## 3. Marginal Distortions

The issue that has bedeviled the field, and has resonated with the concept of a curvilinear aspect to perspective in many analyses over the past six centuries, is the fact that wide-angle pictures constructed in accurate perspective appear, under typical viewing conditions, to be dramatically distorted. Two examples of such distortion are illustrated in Fig. 2, the pinhole camera photograph by Maurice Pirenne (1955) on the roof of the church of San Ignazio in Rome, and a diagrammatic drawing of a courtyard from Vredeman de Vries (1605). The perceived marginal distortions of such images under typical viewing conditions are generally agreed (Field, 2005). But a key question is whether the marginal distortions are some kind of limitation of geometric perspective *per se*, or are due to some defect in the viewing conditions. In the minds (and writings) of many (e.g., Hansen, 1973; Hauck, 1879; Heelan, 1983), these perceived distortions imply a limitation in the logic of geometric perspective; if linear perspective appears distorted, the reasoning goes, then the requirement to achieve undistorted perspective must involve some curvature of the straight lines. Hence, true (undistorted) perspective must incorporate a curvilinear spatial metric of some kind.

This argument, however, is based on the false premise that the perspective geometry in images such as those of Fig. 2 is correct for the typical viewing conditions (because it was correct under the ‘viewing’ conditions of their construction). The parenthetic justification is indubitably true for Pirenne’s pinhole camera image, in particular, because a pinhole eliminates the possible sources of distortion to which a lens is subject and generates a perfectly accurate perspective projection, so it cannot be optically distorted and we must seek elsewhere for the source of perceived distortion. It is perhaps by now obvious that the source of the distortion lies in the viewing conditions, which remain unspecified by Pirenne, Gombrich, Panofsky, Field and many other writers on the topic of marginal distortions.

Indeed, the very term ‘marginal distortions’ betrays a disregard for the basic operation of perspective. By definition, all shapes that are not parallel with the picture plane are depicted as distorted under the perspective transformation. Marginal shapes are distorted because they are viewed at an extreme angle to the line of sight, but they no more or less distorted (or foreshortened) than any other, non-marginal shapes at an oblique angle to the line of sight, so the marginal ones should not be singled out as ‘distortions’. All foreshortening derives from situations where the shape is viewed non-perpendicularly to the line of sight, and the marginal viewing conditions are merely one example of such a situation.

To avoid perceived distortion, the viewing location for accurate geometric perspective should be centered in front of the picture’s vanishing point at the precise distance from which the perspective was constructed (or the distance of the pinhole from the film plane scaled to the enlargement size, in the case of the camera). Any other viewing location violates the requirement that the perspective is geometrically correct when it reaches the retina.

However, the required viewing distance is very far from what is possible for the human eye, for Fig. 2, as for printed illustrations of such demonstrations of marginal distortion in general. To view them correctly, the eye should be about 5 cm from the page, which is obviously very much closer than the physiologically achievable range. To put it another way, for the perspective of these figures to be correct at the typical page viewing distance of about half a meter, each image in Fig. 2 would need to be blown up to about 1 m in width, roughly the size of a typical gallery painting. Viewed under these conditions, the obliqueness of view of the marginal objects would exactly correct the distortion that is seen under the distant viewing conditions. This is what it means for the perspective to be correct for a stable observer at this viewing distance.

Indeed, the perceived distortion was quantified for the de Vries image of Fig. 2B in a study by Todorović (2008). He found that the angles of rotation of up to about 60° from the vertical were perceived as about half that value when the image was viewed at twice the distance of his estimate of the geometric center of projection (CoP), and were reduced by about 80% when viewing from the estimated CoP itself. This result supports the existence of a profound effect of deviations from the CoP on the perception of pictorial perspective. The failure to achieve complete compensation for the rotation distortion was attributed to possible visible surface cues, but could also be due to the fact that the image was presented at a 45° vertical slant rather than frontoparallel in this experiment, which meant that the eye was significantly away from the true CoP, and hence that full compensation should not have been expected.

Incidentally, as Leonardo da Vinci has argued (Richter, 1939, folio 38, sect. 86), the three-dimensionality implied by the perspective is only properly appreciated under these veridical viewing conditions (Tyler, 1998); our brains are so well-attuned to perspective cues that, when we do take the trouble to match our viewing location to the precise CoP, the picture attains a degree of perceived depth comparable to that of the powerful binocular disparity cue. This sense of vivid, realistic depth pops in to perception in a very narrow sphere of space right around the geometrically-defined CoP for each picture. At only a small distance away in any direction, the picture tends to flatten out to 2D percept of the lines on the page, with a much-reduced sense of the depth carried by the perspective construction.

This observation of the vivid sense of depth is counter to the concept of the robustness of perspective (as proposed by Kubovy, 1986), which implies that perspective scenes are appreciated veridically even when viewed from locations away from the CoP. If the viewer's perception were robust to the viewing angle then the shape would not appear foreshortened as the angle of view departed from the CoP. If such compensation failed, the perceived distortion would match the compression of the retinal image.

## 4. Experimental Studies

I am not aware of any experiments having been done in terms of quantifying the perceived depth from perspective, *per se*, but comparable studies have been done for the perceived shape distortion (or perceived foreshortening) of pictorial objects as a function of the angle of view to the picture. One such study, by Vishnawath *et al.* (2005), employed computer-generated scenes (from stationary locations) by under three conditions: with full-cue binocular viewing, with full-cue monocular viewing, and with monocular viewing through an aperture that excluded any information about the surface orientation, such as the frame configuration. Despite the title of their paper ("Why pictures look right when viewed from the wrong place"), they found full compensation under only very limited conditions. Under the *monocular aperture* condition compensation for the viewing angle was minimal, and there was only about 30% compensation under *monocular full-cue* condition. Under the *binocular full-cue* condition they did find full compensation for small angles around the CoP, but the compensation rapidly failed beyond a 30° viewing angle. Thus, the robustness of perspective seemed to operate only when the picture surface was strongly defined by binocular disparity cues, which may seem paradoxical because those surface cues are antagonistic to the perceived depth from the pictorial cues. Similar conclusions were reached by Yang and Kubovy, 1999, and Todorović (2005, 2008). These analyses therefore support the conclusion that objects in pictures appear distorted roughly in proportion to the deviation from the geometrically correct viewing location, particularly when viewed under conditions that optimize the perceived depth in the picture.

For the present purpose, however, it is more germane to note that the perceived depth is usually much reduced in comparison with the implied depth of the perspective construction, which matches my own experience of viewing pictures in galleries. That is to say, robustness of perspective does not hold under conditions approximating those of typical gallery viewing. This paucity of perceived depth can be readily verified (as I have often done) by assessing the most likely center of projection for the perspective construction employed and viewing the work from that location in space with one eye closed. In most cases, under these viewing conditions the depth expands out by an order of magnitude to something approaching the veridical depth perception of a real scene, again validating that robustness of perspective was not in force under the original viewing conditions.

Another study focuses on the issue of natural perspective, in terms of the perception of the rays converging on the eye. Koenderink *et al.* (2010) make the strong statement that the visual rays emanating geometrically from the eye are perceived not as a divergent fan but as parallel with the principal ray from the fovea. Moreover, the implication is, lines that are parallel and receding from us in physical space should appear to be convergent, in a physical form of the perspective convergence in pictures.

This claim is strongly empirically based, but it is nevertheless logically absurd in several respects, including some in their own publications. For example, if the rays are parallel, then all objects between a pair of rays (i.e., subtending the same visual angle) would be perceived as the same size, in violation of Emmert's Law that the perceived size of a given visual angle is proportional to distance from the viewer.

A second implication of the parallel perception of diverging visual rays is that planes orthogonal to the visual rays at different angular direction should appear coplanar, and that if the eye is located at the center of a sphere, the surface of that sphere should appear to be a flat plane. Moreover, a large frontoparallel surface should appear to curve away from us in a hyperbolic 'fisheye' distortion forming a curved surface. Both predictions are indeed the case for snapshots of a scene from a rotating eye or camera pointing in different angular directions. A spherical surface always appears frontoparallel in the direction of gaze from its center, and a flat surface forms a hyperbolic set of angles as the direction of gaze is swept past it. But this moving-eye analysis is not the assumption of the Koenderink *et al.* (2010) paper, which is discussing the pictorial space of flat pictures. When viewing a flat wall, or a picture of the frontal plane from one visual direction, few dispute that it appears flat rather than curved to any appreciable degree.

On the other hand, even in the full 3D world we do experience some tendency in the direction of the parallelism of Koenderink *et al.* (2010), in that a long corridor with parallel sides does appear to have some convergence of its parallels. This is a puzzle in a world of fully adapted organisms, since they (we) need to interact with space as veridically as possible, but although we 'know' that the corridor is in fact constructed with parallel sides, when asked how it looks we report that the sides appear as converging to some extent (though far less than predicted by the strict principle of the parallelity of the visual rays).

## 5. Allowable Viewing Conditions

Considering these large-scale viewing conditions, however, brings up the issue of the limits of the stationary observer. Does the restriction to the CoP mean that the eye must be absolutely stationary, or merely that it needs to adhere to a stable location while having freedom to rotate around that location? The logic of the perspective construction, based on projection of the scene through the picture plane to a point, implies the latter. If the optic array impinging on that point from the 2D picture exactly reproduces the optic array generated by the original 3D scene, the information in that optic array is independent of orientation of the eye receiving that information. As long as the pupil (strictly, the optical center of the eye (see Note 5)) remains at the CoP, the eye is free to look in any direction whatever and still receive the correct information about the scene from its perspective construction. This orientation independence follows from the fact that the perspective construction is defined at a point in space, without reference to any particular projection lines, and hence if the pupil is placed at this location, the optic information will match that of the original scene regardless of the orientation of the eyeball. This is a freedom that is rarely granted to the logic of the correct perspective construction (see Note 6).

Instead of requiring the correct viewing location, most analyses (such as the following examples of Leonardo da Vinci and Piero della Francesca), simply assume that the picture will be viewed from a different distance than that implied in its construction (White, 1957), and recommend strategies to minimize the resultant distortion (without apparently recognizing its source). A typical

Renaissance picture, for example, is constructed with the distance points of the one-point construction set at the left and right edges of the frame, implying a required viewing distance of exactly half the frame width. Since gallery pictures are typically viewed at a distance of about twice the frame width or more, this construction implies that they are typically viewed around four times further away than they should be.

Thus, my assessment is that, historically, most analysts talk rather loosely about the issue of the field of view and marginal distortions. Piero della Francesca (1482), for example, has a diagram in *De Perspectiva Pingendi* indicating that marginal distortions become unacceptable beyond a 45° angle, and Leonardo da Vinci recommends restricting the width to distance ratio to less that 1:10 for the same reason. But the fact is that geometrically correct perspective *requires* marginal distortions, as amply demonstrated by Pirenne (1970) and White (1957). The distortions of the image are required to correct the inevitable compression from the oblique viewing angle of the extreme parts of the viewed plane. When viewed from the correct CoP, the marginal distortions generate the unambiguously *correct* reproduction of the original scene at the (single) eye of the viewer.

The problem arises when there are many observers, or two eyes, or a moving observer, and so on. Now the geometric convention breaks down, and the correct projection is viewed as a marginal distortion that needs, as Leonardo and Piero were fully aware, to be avoided. Thus, there is literally no way to paint a picture in correct perspective for its natural viewing environment, which is a room full of moving observers. Strictly speaking, correct perspective is an insoluble problem for the full viewing environment, although these problems may be avoided by minimizing the geometric cues to perspective that commit to a particular CoP, and relying on the less definitive distance cues, such as the familiar size of generic objects and aerial perspective (of which the most famous example may be the background landscape of Leonardo da Vinci's 'Mona Lisa', but is very much the case for an endless series of subsequent landscapes of only partially familiar generic objects).

## 6. Analysis of Curvilinear Geometry

Leonardo da Vinci made the distinction between natural and artificial perspective, and, although it is difficult to divine his exact meaning, it has been interpreted that at least one of his (unspecified) number of forms of perspective involves a spherical surface everywhere equidistant from the eye (White, 1957, pp. 2–3). Johannes Kepler follows the same line of thought:

> "Our vision does not in fact have a plane surface like a tablet, on which it contemplates the painting of a half-sphere, but rather that image of the sky, against

which it sees comets, it produces in itself *as spherical by natural instinct of vision*. And if the image of objects is projected into a concave sphere with straight lines of extension, the representations of those lines will not be straight, but in fact curved, just as in the circle, no doubt, of the greatest sphere, if it is seen from the center, as we teach about projection from astrolabes."

Johannes Kepler (∼1625, *Appendix Hyperaspistes*, my emphasis)

Kepler (1625) thus raises the issue of the natural form of perspective being inherently curved, which seems to have informed a line of thinking of subsequent generations of perspectivists. Two things need to be said about this concept, however. One is that the notion of a spherical projection surface must be parsed as mystical rather than physical, since there is no optical or physiological entity corresponding to the spherical surface produced by Kepler's 'natural instinct of vision'. It seems to be a concept that anticipates Gogel's 'equidistance tendency' (Gogel and Tietz, 1973), that objects devoid of depth cues (such as points of light in the dark) tend to be seen at a fixed egocentric distance in space, which would form a spherical surface if projected at all possible angles. Kepler takes this idea to an extreme of realization in envisaging objects, presumably drawn from the real world, being projected back onto this distance conceived as a spherical projection screen as the basis of natural perspective.

The other aspect that needs examining is the idea that such an imaginary projection would in fact convert straight lines into (great circle) curves. Although this concept is geometrically valid, from the viewpoint of the projecting agent the curvature would be a purely depth curvature in terms of egocentric distance, and would only be visible in the third dimension. On the retina of the viewer performing the projection (and hence on the canvas being viewed by that retina for the perspective rendition), the lines would remain just as straight as before.

Gombrich's (1974, pp. 89–90) analysis of the situation is as follows:

"Gibson has warned us that the night sky is not the case with which to begin the analysis of stimulus information, but what about ending it in this way, or rather regarding it as a limiting case in every sense of the term?

. . .

"Why, then, do we not see a flat expanse but a convex field? Because — my answer would have to be — we can never separate the static view from the flow of information that precedes and follows it.

"If the apparent vault is really composed of a succession of narrow fields of vision at right angles to the momentary line of sight, we must not only expect such a curvature while we scan the heavens, we are bound also to perceive it while we try to keep our eyes fixed on one point in the sky."

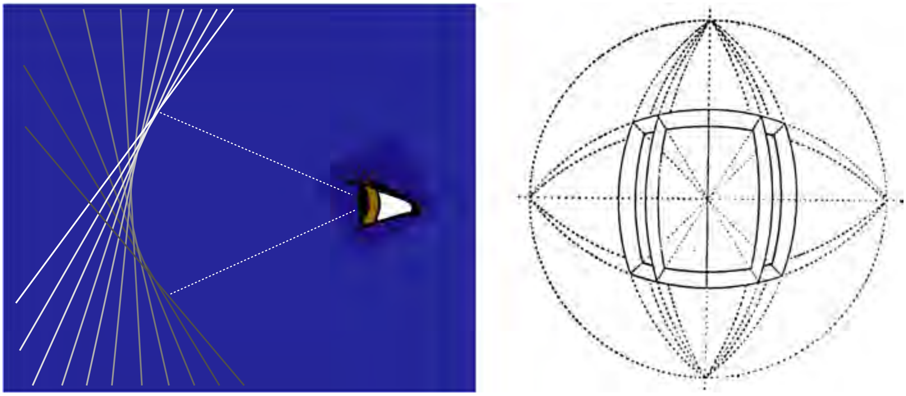

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**Figure 3.** (A) Illustration of the curve generated by an integrated view of a series of straight line projections. (B) Illustration of the curvilinear perspective of a 180° view of the canonical cube by John White (1957). This figure is published in color in the online version.

In other words, Gombrich is proposing that our attention is focused at any moment not on the whole field but on the region of the high-resolution fovea around the line of sight. He develops this idea in the context of his observation of the vapor trails quoted at the beginning of this article. His concept of the generation of curvature by the integration of a succession of straight lines changing angle over time may be captured by the concept of an optical coma, in which the straight lines of light reflected by a curved mirror sum to form a curved boundary.

The curved boundary of the assemblage of straight lines is illustrated in Fig. 3A, where the succession of greys is intended to indicate the temporal progression of the straight vapour trails as our fixation moves across the sky. The straight lines integrate into a curved boundary and Gombrich is arguing that this curved boundary is a valid concept when the movement of the eye is allowed as an aspect of the experience of 'natural perspective'. In this sense, then, artists attempting to depict the sense of space conveyed in natural perspective are justified in using curved lines to capture the experience. It is this concept of curved lines connecting opposing vanishing points (Gombrich, 1974) that was developed by his associate, John White, to depict the wide-angle or 'fish-eye' view of a 180° scene reproduced in Fig. 3B (White, 1957, Fig. 8, p. 208).

More recently, this basic approach has formed a continuing theme in the works of David Hockney from his wide-angle photocollages to paintings such as his 'A Visit with Christopher and Don, Santa Monica Canyon, 1984' (1985) and 'Caribbean Tea Time' folding screen (1987). Similarly, Panofsky (1927) tried to build up a uniform solution for perspective curvature based on ideas from Vitruvius and da Vinci, which may have some perceptual resonance as delineated by White, but I will argue that this approach is no more 'correct' than is linear perspective for the problem of the natural viewing environment.

It should be noted that White's diagram for the projection of the canonical cube has five vanishing points, one at the center and one in each of the cardinal directions. In this sense, it has gone beyond the classic concepts of one-, two- and three-point perspective construction for the canonical cube (see Tyler, 1998) to allow a non-Euclidean 'fish-eye' projection of the world that we may term 'five-point perspective'. White's construction is the special case of wide-angle perspective where one of the vanishing points is directly ahead of us. If the same cube is projected obliquely onto the picture plane (in terms of a horizontal rotation), the construction will reduce its point count to become a form of four-point perspective (a paradoxical reduction, in the sense that the Euclidean one-point perspective *increases* to two-point perspective when viewed obliquely). Thus, if we allow the fish-eye projection of 180° to the plane, we can add four- and five-point schemes to the traditional one-, two- and three-point perspective definitions (i.e., the projections of *orthogonal* vanishing points enforced by the canonical cube). Mauritz Escher has used this approach with his 'Up and Down' (1947) and 'House of Stairs' (1951) engravings, both of which are based on a vertically-oriented version of the wide-angle perspective scheme incorporating 180° and 270° perspective views, respectively (Ernst, 1976, pp. 52–55), within the vertical height of the picture (as compared with a typical vertical angle of view of about 30° under normal viewing conditions). However, by using oblique perspective in the orthogonal direction and extending the angle beyond 180° (as in the 'Up and Down' engraving), Escher opens up the scheme to imply the possibility of continuation to a 360° spherical projection and beyond. In other words, once the perspective scheme wraps to 360° in the picture plane, it can continue indefinitely, limited only by the size of the projection plane chosen for the work.

Consequently, the definition of the number of vanishing points depends solely on the scale of the mapping relative to the canvas size, forming an ∞-point perspective scheme. This ∞-point scheme is the logical conclusion of the extension of strict linear perspective to time-integrated curved representations. In principle, we can explore the world in the full 360° of the spherical coordinates in all directions, so the ultimate scope of what should be reproduced to convey the full sphere of vision is the ∞-point perspective scheme. Other artists have pursued different approaches to the concept of a 360° view, such as Kiki Smith's 360° self-portrait 'My Blue Lake' (1995) reproduced in Fig. 4. In such cases, the concept is the reciprocal one of a 360° encircling view of a defined object rather than a panoramic view around a point in space, indicating that further explorations are possible.

**Figure 4.** 'My Blue Lake' by Kiki Smith (1995). This figure is published in color in the online version.

At this point, it should have become clear that *all* uses of perspective are compositional conventions (with the exception of the 'perspective box', which physically constrains the viewing position to the geometric center of projection). "We do not always realize," wrote Sir Herbert Read, "that the theory of perspective developed in the fifteenth century is a scientific convention; it is merely one way of describing space and has no absolute validity." (cited in Gombrich, 1960, p. 247). If there is no 'correct' solution for the natural viewing environment, any attempt to convey depth in the scene has to make some compromise. What is interesting is that, historically, different compromises predominated in different eras — local perspective in the Mediaeval era; accurate one-point perspective in the *quattrocento* (see Note 7) (with forays into curvilinear perspective); forms-in-space in the late Renaissance, or Mannerist era; the widespread use of oblique (or two-point) perspective in the 18th–19th centuries; analytic cubism in the early 20th century, and so on. To use Gombrich's term, each era adopted a predominant *schema* for the use of perspective that accompanied other stylistic conventions (although such characterizations are always broad generalizations with local exceptions). Each era took a different 'slant' on the problem, but each should be viewed as a convention in the sense that Panofsky seems to have intended. His term 'symbolic form' may be rather overladen, but it may be interpreted as meaning a *compositional con-*

*vention*, a schema for depicting some aspects of space on the 2D surface when it cannot all be depicted there (Gombrich, 1972).

In this sense, the predominant perspective schema at any given time represents a kind of artistic Kuhnian paradigm of how artists were thinking in that era. To give one example, in the Renaissance geometry seems to have been typically thought of as ‘science’, both in the sense of classical truth embodied in the treatises of Euclid and Vitruvius, in particular, and also futuristically as the artistic way forward from Mediaeval conventions. We now think of perspective as mathematics rather than empirical science, but it is one expression of the spirit of that time, and in the *quattrocento* that expression was very geometric. Michelangelo can be viewed as the individual most responsible for changing this approach and for bringing in a new ethos of emotional power in which linear perspective almost disappeared and was largely replaced in the *cinquecento* by human forms in space, as epitomized by his ‘Last Judgement’. But this new schema is still a depiction of three-dimensional space, another form of perspective, although it uses a different set of conventions to achieve this three-dimensionality, and one that does not require straight lines or the specification of an explicit vanishing point. Hence, this *cinquecento* transition illustrates how the ‘symbolic form’ of perspective, in the restricted sense of generally accepted conventions of spatial representation, kept changing over time.

The difference between a schema and an exploration is illustrated by the few examples of curvilinear perspective that are known from the Renaissance. Interest was apparently sparked by the construction of curved mirrors in the early *quattrocento* in the Netherlands, of which the first and best-known depiction is the one by Jan van Eyck in the center of the ‘Arnolfini Double Portrait’ (1434). The mirror shows the back half of the room occupied by two men, presumably van Eyck and his brother Hubert (Fig. 5). On the one hand the

**Figure 5.** ‘Arnolfini Double Portrait’ by Jan van Eyck (1434), with details of the spherical mirror surrounded by the stations of the cross below the artist’s signature, and of the distorted reflection in the mirror. This figure is published in color in the online version.

spherical mirror can be taken as the literal depiction of an expensive object owned by the rich Italian merchant, while on the other hand it also has the symbolic meaning of the eye of the painter (or of God) surveying the scene before it, the first amplified by his signature immediately above it, the second by the Stations of the Cross shown encircling the mirror (Alpers, 1977).

However, this mirror has the third role of introducing the concept of curvilinear perspective space to the Renaissance eye. Although used a few other times (notably by Petrus Christus in his 'St Eligius', 1469), curved mirrors did not become a popular theme of the time. Only a handful are depicted throughout the *quattrocento*. Nevertheless, they may have been used as a tool to conceptualize the forms of curvilinear perspective that are seen in the miniature of a procession by Jean Fouquet ('Arrival of Emperor Charles IV at the Basilica St Denis in 1378', c. 1460) and the 'Crucifixion' by Andrea Mantegna (c. 1460). These two paintings (Fig. 6) both have curvilinear *pavimenti* that go beyond what might be expected for the physical structure. This exploratory use of curvilinear perspective depictions, though interesting as an advanced conceptualization, clearly did not take hold as a dominant theme in the way that linear perspective had, since they remained isolated examples in a era dominated by one-point perspective (White, 1957).

A recent elaboration of curvilinear perspective was introduced in the 20th century by Dick Termes in his lifelong interest in perspective projection to the sphere. As Termes (1991) points out, the natural projection of a cube to an enveloping sphere is six-point perspective, because each of the three sets of orthogonal parallels projects to a vanishing point in two opposing directions. The ultimate expression of the 'Sphere of Vision' of Vitruvius and Bosse (Fig. 1) is, indeed, the eye centered in a sphere of a defined diameter. If the outside world is projected onto this sphere instead of the usual approach of projecting

A B

**Figure 6.** (A) 'Arrival of Emperor Charles IV at the Basilica St Denis in 1378' by Jean Fouquet (c. 1460). (B) 'The Crucifixion' by Andrea Mantegna (c. 1460). This figure is published in color in the online version.

it onto the picture plane, the perspective construction is inevitably the six-point scheme. However, it should be emphasized that, unlike the flat picture plane of the one-, two-, three-, four- and five-point schemes discussed so far, the six-point construction is specific to a spherical 'canvas' (or any other enclosing shape, such as the cube that is used in 'virtual reality' projections exemplified by the CAVE; Cruz-Neira *et al.*, 1992) and does not apply to the flat canvas. It should also be clear that the six-point scheme is, like any perspective scheme, specific to the viewing location. Thus, if Termes' spherical constructions were viewed from their center of projection inside the sphere, they could be considered veridical perspective constructions. However, since they are in fact viewed from *outside* the surface of the sphere, the six-point scheme is no longer a visually correct projection, and is a perspective convention like any other.

## 7. Conclusion

This survey has considered a number of forms of curvilinear perspective, both on the standard picture plane and on curved surfaces. Consideration of these projections by no means exhausts the range of possible forms of curvilinear perspective, but has attempted to give some insight into the relevant considerations. In summary, the only valid form of perspective for the flat canvas is linear perspective, as characterized by the one-, two- and three-point perspective schemes for scenes consisting of orthogonal sets of parallels such as the canonical cube (and, of course, by the corresponding projections of curved objects), and it is valid only for a viewing location at the geometric center of projection for which it was constructed. Viewing from any other location (particularly in the case of wide-field images viewed from greater than the required distance) is susceptible to perceived distortions that have often been misinterpreted to imply that linear perspective is inadequate and that some form of curvilinear perspective would be more representative. However, though it may be valid for any particular view, no form of perspective projection can capture the integrated experience of space viewed over time that constitutes the full concept of the natural perspective of the moving observer. If this is the goal, it can only be approximated by some perspective convention, any form of which has major deficiencies that cannot be overcome to provide a complete experience of natural perspective. In fact, the most compelling form of natural perspective is achieved by viewing large-scale, high-resolution linear perspective images from their geometric center of projection. As long the geometric perspective is viewed with one eye and with that eye held at one location (both restrictions for any form of static perspective scheme), the perceptual experience is of a full, explorable 3D space, which cannot be said of any other form of geometric perspective convention.

## Notes

1. Authors who describe Renaissance perspective theory and practice without mentioning the contribution of Vitruvius included White (1957); Wittkower and Carter (1953); Ivins (1975); Harries (2001); and Field (2005). A classic example is Kemp (1990, p. 9), who states "Linear perspective was invented by Filippo Brunelleschi. His priority has never been seriously questioned, either at the time or subsequently, although we can be sure it would have been if anyone had even flimsy grounds for an alternative claim." Panofsky (1924–5, 1991) discusses Vitruvius but explicitly repudiates the interpretation of 'circinique centrum' as 'vanishing point', suggesting that it may have meant the center of projection instead. White (1957), on the other hand, strongly supports it, saying that it is "very difficult to believe, on and rules of evidence, that in the [. . . ] case of the phrase 'ad circini centrum omnium linearum responsus', anything can be meant except the direct convergence of all lines towards a single point." Edgerton (1975) is also one of the few to give Vitruvius appropriate recognition for his perspective concepts, but he does not discuss their transmission to the Renaissance practitioners.
2. "*De Architectura* was not unknown in the Middle Ages, but its corrupt Greek vocabulary was an insurmountable obstacle for most readers, who were mystified by such aesthetic terms as *eurythmia* and *simmetria*. [In the 14th century,] Petrarch annotated his manuscript of Vitruvius with explanatory notes. *De Architectura* was quoted by Boccaccio, who read the text with fascination, and by Benvenuto da Imola. In 1414 Poggio Bracciolini, who recovered many manuscripts of classical authors, became interested in the work after finding a copy in the Swiss monastery of S. Gall. In the 1420s, Cardinal Branda Castiglione erected a group of buildings in the country village of Castiglione Olona, near Varese. The Cardinal and his architect drew on Vitruvius and Pliny. Although Vitruvius' writing was obscure, sometimes nearly unintelligible, his fame started to spread." (Encyclopedia Brittannica, 1911).
3. Alberti refers extensively to Vitruvius in his "*De re aedificatoria*" (c. 1450), as does Ghiberti in his *Commentarii* (c. 1447–1455). Field (2005, pp. 74–76) argues that Piero della Francesca was acquainted with Vitruvius' text. Leonardo da Vinci is famous for his diagram of the "Vitruvian Man", whose proportions in relation to the circumscribing circles and square are explicitly specified in Vitruvius Book III, I, 2–3 (trans. Granger, 1931): "The close adherence of Leonardo's diagram to these proportions implies, therefore, that he must have scrutinized Vitruvius' text in detail."

4. Assuming that the plane was flying on a steady course, the vapour trail would follow a great circle around the globe, and hence would have the corresponding curvature, but at a typical altitude of six miles a viewing field of, say, 45° would imply that the vapour trail had an arc length of 12 miles, or 1/2000 of the earth's circumference (i.e., a subtense of only ∼10 arc minutes) and should be regarded as straight to within the limits of human discrimination.

5. The optical center is the point in a compound lens system through which rays impinging on the lens emerge at the same angle on average.

6. As for example in Field (2005, p. 153) "It is, of course, assumed that the eye will not move: perspective deals with the geometry of a single glance." No argument is given to support this statement, which is valid up to the colon, but as my preceding logic shows, the restriction against movement is not applicable to movements that leave the pupil at the CoP (or, approximately, to rotational movements of multiple glances from this location).

7. Strictly, use of the accurate one-point construction began in the 1300s, with examples of accurate convergence of floor tiles, taking hold in the early 1400s and fading dramatically in the early 1500s, led by the Italians. Partially accurate central convergence is, of course, known back to Roman times as implied by the analysis of Vitruvius in the Introduction.

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