A Horopter for Two-Point Perspective

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ABSTRACT

Linear perspective is constructed for a particular viewing location with respect to the scene being viewed and, importantly, the location of the canvas between the viewer and the scene. Conversely, both the scene and the center of projection may be reconstructed with some knowledge of the structure of the scene. For example, if it is known that the objects depicted have symmetrical features, such as equiangular corners, the center of projection is constrained to a single line (or point) in space. For one-point perspective (with a single vanishing point for all lines that are not parallel to the canvas plane), the constraint line runs from the vanishing point perpendicular to the canvas. For two-point perspective, in which the objects depicted are oblique to the canvas, the constraint line is a semicircle joining the two vanishing points. A viewer located at any point on the circle will see the depicted objects as rectangular and symmetric, and will have no grounds for knowing that the perspective was not constructed for this viewing location (unless there are objects that are known to be square, i.e., a further symmetry constraint on the object structures). This semi-circular line of rectangular validity forms a kind of horopter for two-point perspective. Moving around this semi-circular line for an architectural scene gives the viewer the odd impression of the architecture reforming itself in credible fashion to form an array of equally plausible structures.

1. BASIC RULES OF LINEAR PERSPECTIVE

The rules of perspective have been known for centuries, but it is worth reiterating a few of them as background for the development of the new concept of a constraint line, or horopter, within the perspective domain. The domain of linear perspective is the geometry of projection of the lines in a scene through a picture plane to a point in space corresponding to the pupil of the viewing eye (Fig. 1). The picture plane would correspond to the canvas on which the painter wishes to depict the scene. For correct perspective, the picture will generate the same arrangement of light rays at the eye as did the scene behind it. When viewed from this point in space, therefore, the picture will form exactly the same image on the retina as did the original scene. The different forms of perspective construction concern the rules that apply to specific structures, but all are subcases of the same optical transform. It is important to be clear that the laws of perspective projection have no relation to processes within the eye or the brain, but relate solely to the optical information available at the eye from the projection plane.

Fig. 1. Projection of parallel lines in space. Three parallel lines in the scene at left are projected through a rectangular picture plane to the point where the observer’s eye is located. The light rays projecting to the eye are shown by dashed lines. The three parallel lines project as straight lines in the picture, but not as parallel because their orientation is not parallel to the picture plane. The point in the picture where the projected lines converge is termed the vanishing point (VP).
The eye forms the center of projection for the perspective construction. The center of projection is a physical location in front of the picture plane for which the perspective provides the veridical information for the three-dimensional scene. In general, viewing the picture from any other location generates a distorted projection corresponding to a different visual scene.

All straight lines in space project to straight lines (or points, in the limit) in the picture plane. This fact is a simple consequence of the geometry of projection through a point in space (corresponding to projection through the pupil of one eye). If a line is parallel to the picture plane, it must project to a straight line on the picture plane by virtue of the similar triangles formed with their apex at the eye. Obviously, rotating the line within the plane of projection will not introduce any curvature, just a change in its extent within the line of projection. In the limit, the projected line may contract to a point in the picture plane when the line is viewed head on. Lines of any orientation can be described by this construction. Thus, all such point projections of straight lines are to straight lines or points. There are no cases in which the perspective transform through a point introduces curvature of straight lines.

The projections of all sets of parallels in space meet at their mutual vanishing point somewhere in the picture plane (Fig. 1). Each different set of parallel lines ends at a different vanishing point (which may be at infinity in the special case of sets that are parallel to the picture plane). This common vanishing point applies to all lines parallel to a particular line, regardless of where in the visual field the lines arise. Thus, the first job in perspective projection is to identify all the lines in the scene that are parallel to each other, then make sure that they are drawn so as to project to a common vanishing point.

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**Fig. 2A.** A painter at his canvas painting a scene of a rectangular grid with ziggurat. Dashed lines in the grid are parallel to the canvas (transversals), solid lines are perpendicular to it. **B.** Depiction of the scene as projecting to the plane of the canvas, with the perpendicular lines converging at the central vanishing point, while the transversals remain parallel to it.
Parallel lines in space that are also parallel to the picture plane remain parallel to each other in the projection. This leads to the particular case of one-point perspective, in which all the lines on the scene are either parallel with the line of sight or at right angles to it, parallel with the picture plane. Such lines are illustrated in the overhead view of the artist before his canvas in Fig. 2A. The scene consists of a pyramid structure on a tiled surface. The edges of the tiles that are parallel to the picture plane are shown as dashed lines, and will project as parallel in the picture. The edges that are at right angles to the picture plane are shown as solid lines, and will all project as converging to the central vanishing point in the picture. The same rules are apply to the lines in the ziggurat, although they are all shown solid for the 3D effect. The resulting picture is shown in Fig. 2B.

It is important to note that, if the vanishing point in central perspective is displaced from the center of view, the simplicity of the central perspective construction has been violated and a second vanishing point arises at 90º from this displaced vanishing point (for lines in the same plane). Each displacement implies a different location for the orthogonal vanishing point, aligned diametrically opposite the direction of the displacement.

2. THE IMPORTANCE OF ANGLE OF VIEW
The viewer’s angle to any pair of vanishing points is the same as the angle between the generating parallels in space, regardless of their angle of projection in the picture. In particular, the vanishing points for any 90º angle in space form a 90º angle at the viewer's eye (Fig. 4). This result may be seen by considering the members of their respective parallel bundles heading directly toward the viewer's eye. These lines at the eye form the same angle as any other pair from the two bundles. The angle between these lines at the eye, and hence the viewing angle between the vanishing points, therefore matches the angle of the lines in space.

![Diagram](image_url)

Fig. 3. For objects viewed at 45º, the viewer’s angle to the vanishing points is 90º, matching the angle between the edges of the cube that generated the vanishing points.
A classic case of the 90° rule is the diagonals of any square, which are always at 90° to each other. The vanishing points (or "distance points", da Vinci, 1492) for these diagonals should therefore form a 90° angle at the center of projection, regardless of their orientation in space. Rotate the angle in any direction whatever in three-dimensional space (even to the point of complete foreshortening) and the vanishing points will nonetheless hold to a strict 90° angle at the viewer’s eye. In terms of pictorial distance, this angle between the vanishing points corresponds to the same distance in the picture plane (except for the tan transform for projection of the equal angles at the eye onto the plane).

3. PERSPECTIVE DEPTH PERCEPTION, MARGINAL DISTORTIONS, AND STEREOSCOPIC PERSPECTIVE.

A particular case of perspective distortion is the marginal distortions illustrated by the well-known courtyard scene constructed by Vredemann de Vries. A glance at this figure gives the impression that the obelisk-shaped pillars, in particular, are strongly splayed out and obliquely skewed. Yet, paradoxically, this figure was constructed according to the precise laws of perspective. It is, in fact, predominantly a two-point perspective construction, with the two vanishing points arranged vertically one above the other at a separation of about 14 cm, implying that the center of projection is 7 cm (~3 inches) away from the page, in front of the white disk overlaid at the center of the picture. In order to view the perspective veridically according to its construction, therefore, the picture should be viewed at this close distance, a feat that is impossible for all but the strongest myopes.

Fig. 4. Extreme perspective depiction of a courtyard, by Vredemann de Vries (1605).

If one makes a copy of this figure and expands to a point where the center of projection is accessible, a remarkable observation is made. When viewed from the center of projection, all the marginal distortions appear to evaporate. The splay is corrected and the obelisk pillars are seen as symmetrical on rectangular bases, all pointing in the same direction. This should not be surprising because the perspective was constructed for symmetrical pillars, so when viewed from the
center of projection, it provides the accurate information for such symmetry. It is only the fact that the construction is viewed from the wrong distance that introduces the perceptual distortion. Renaissance artists were well aware of this issue in terms of the distance points of projection of the corners of squares in the pavimento. The physical distance between the vanishing points depends on the intended viewing distance, but a good rule of thumb is that it should correspond to a distance of at least twice the width of the picture. Leonardo da Vinci recommended that the viewing distance should be at least 20 times the height of the largest objects depicted.

An even more remarkable transformation that occurs when the eye is correctly placed is that the perspective generates a profound sense of perceived depth. Viewed at the normal distance, most people perceive the image as generating distorted architecture with a depth of a few centimeters at most, if asked to imagine the paper cut-out behind the page that would give rise to this image (in the fashion of the pop-up greeting cards and children’s books). Viewed correctly, however, the sense of depth and three-dimensionality becomes profound, extending several feet behind the page, and viewable at all angles of regard (as long as the eye remains at the center of projection). This effect may be termed stereoscopic perspective, a sense of three-dimensionality as strong and compelling as that generated by binocular stereopsis. Binocular cues are often regarded as the benchmark of the depth sense, against which all other depth cues should be calibrated. It was a revelation to find that pictorial cues could generate as strong a sense of depth as binocular disparity itself, since one is so used to regarding perspective as an interpretive cue of the projective distortion onto a flat surface. It is only when viewed with a large visual angle with the structure projecting veridically onto the retina that the equivalent sense of depth is obtained from the perspective cues. It raises the question of the remarkable “perspective engine” that must exist somewhere in the neural processing (most likely in the parietal lobe) to extract the depth interpretation from the variety of intersecting angles and figural features. The key point is that we do not just understand in a cognitive sense the construction of the depicted objects, we actually develop a full three-dimension percept of their relationships in the constructed space before us. It is the immediacy of the perceived space that highlights the sophistication of the neural processing involved.

4. PERSPECTIVE DISTORTIONS FROM OFF-AXIS VIEWING

In addition to the marginal distortions that arise from the wrong viewing distance, it has long been known that viewing paintings from the wrong viewing angle generates perceived distortions of the scene geometry. Indeed, it is the description of such distortions from the era of classical Greece that tells us of the accuracy of the perspective of that time. The first mentions of perspective constructions, by Plato and contemporaries in the 5th century BC, were provoked by the dramatic use of perspective by the scene-painter, Agatharchus of Athens in the scenery for the plays of Aeschylus and Sophocles. In fact, the problem of off-axis distortion was evident right at the outset, when Plato inveighs against the distortions generated by viewing the scenery from the seats at the side of the theater. This historical controversy illustrates that any deviation from the true center of projection generates perspective distortions.

There is, however, a restricted sense in which distortions can be avoided for all angular deviations from the center of projection. One of the key signatures of perspective distortion is the deviation of angles from the right angle. A high proportion of objects in the world of human architecture are rectangular. When we view a perspective scene, therefore, we typically expect the major angles of the objects depicted to be right angles, and a perceived deviation from rectangularity is an indication that the perspective is distorted. Indeed, Perkins (1973) developed a set of laws of perspective, the basis of which is that there is a strong tendency to perceive any three-dimensional angle as a right angle (as long as it meets certain geometric constraints). In one-point perspective scenes such as Fig. 2, we as viewers can use the rectangularity rule in order to make any sense of the array of lines before us. One can identify three levels at such constraints are applied.

At the deepest level, we do not have positive information that the lines are straight (in three-dimensional space). They could be a net of threads that are hanging in space in all forms of curves, but just happen to be oriented so as to appear straight from our viewing location. Such a conspiracy of happenstance is so unlikely that we invoke (perceptually) the generic viewpoint assumption (Nakayama & Shimojo, 1992) that they are, in fact, straight lines in 3D. At the next level, the straight lines could form a wide array of structures with of all kinds of angles. The continuity constraint across each intersection implies that the straight segments are themselves aligned into straight lines. Finally, the right-angle constraint could convert any view of a rectangular scene into its veridical percept. If it applied to all angles, therefore,
the right angle constraint could overcome any viewpoint distortion to reconstruct the original scene of rectangular constructions. However, the constraint as enunciated by Perkins (1973) and subsequent publications has more limited application – it applies only to Y-intersection corners and it defines the conditions under which the rectangular constraint does not apply – where it does apply it is a tendency rather than an absolute requirement. In any event, Fig. 4 illustrates that the perceived depth structure does not remain invariant as the eye is moved in front of the image. Distortions of the type depicted by La Gournerie are clearly evident, implying that the perspective construction is not fully corrected by the right-angle constraint. Indeed, such distortions also violate another important constraint – that symmetrical structures are preferred to asymmetric ones (Vetter Poggio & Bülthoff, 1994; Kontsevich, 1996). The shear distortion of Fig. 4 makes the right-hand rear angle acute as the left-hand rear angle becomes obtuse, whereas a symmetric solution would make them equal.

Fig. 4. One-point perspective distortion analysis by Jules de la Gournerie (1859). The three views at left correspond viewing locations in front of the three eye-line points at right.

This analysis illustrates that the understanding of how we see perspective is far from complete. Despite two principles tending to rectangularize La Gournerie’s construction, it is seen with rhomboidal distortion of the ground-plan (Fig. 4). It is not obvious what principle the visual system uses to generate its depth reconstruction. A geometer does it by the global principle of reconstruction of the vanishing points. The series of converging lines from the floor and roof orthogonals converge to the rightmost vanishing point indicated by La Gournerie. This is a global concept because it does not apply to any local line in the image (if La Gournerie’s construction lines are removed). Having constructed this vanishing point, the geometer has the information that it is to the right of the centered viewing point, and therefore that the architectural orthogonals are running obliquely to the right rather than being orthogonal to the canvas. The mechanisms by which the visual system could achieve the distorted reconstruction in the absence of a geometer’s toolkit remain obscure, but it seems to do so with remarkable fidelity. The net result is that, when viewed from locations lateral to the center of projection, the one-point construction is subject to a distorted depth interpretation in
which right angles are sheared to a rhomboidal geometry. The question to be addressed is whether this is a general result for all kinds of perspective, or whether there are conditions under which the right-angle interpretation is preserved.

5. DERIVATION OF THE ANGLE-PRESERVING HOROPTER FOR 2-POINT PERSPECTIVE

The effects of varying the viewing location may now be considered for the case of the two-point perspective construction of Fig. 3, with the vanishing points on the canvas forming a 90° angle at the eye. In the two point construction, all the dihedral angles in the scene are right angles with their edge from the intersection of their constituent planes parallel to the surface of the picture plane. All horizontal lines project to one of the two vanishing points. Conversely, it is generally true that any pair of lines passing through the two vanishing points in the picture project to lines that meet at a right angle in the scene (if they meet at all, i.e., if they are coplanar). That is say, for any viewing location for which the two vanishing points subtend a 90° angle at the eye, all the projected angles should be seen as right angles.

Fig. 5. A. For two-point perspective, the picture is correctly viewed at the distance that matches the span to each vanishing point. B. The vanishing points for two-point perspective remain at a 90° angle for all viewing locations around the circle of Thales.

By the well-known theorem of Thales, the locus for which the lines through two points meet at a right angle is a circle with its diameter through the two points, as depicted in Fig. 3B. From any point on this semi-circle, a viewer will see the two vanishing points with a 90° angle at the eye. Consequently, the locus of these viewing locations all provide the information for an undistorted view of all the right angles in the scene. This locus is reminiscent of the locus of the horopter (or “horizon line”) of zero disparities in binocular vision. In the case of perspective, it is a locus of zero deviation from rectangularity. It may therefore be termed an angle-preserving “horopter” for two-point perspective.

Is the scene perceived as undistorted in general from all points along this locus? It should be clear that the view is only completely undistorted from the original center of the projection. The distortions from other points along the locus are orthogonal shears within the two-point scheme, which are right-angle preserving distortions. That is, square objects in the scene will always project as rectangles with an aspect ratio dependent on the viewing location. The architecture viewed from this locus will, therefore always be valid in terms of a rectangularity constraint, but will form a different geometry from each location along this horopter.

The example of Fig. 6 illustrates that two extreme viewpoints generate perspective images that form perfectly plausible rectangular architecture. The only difference is that the left doorway is wider than the right-hand one from the left viewing position and vice-versa for the right viewing position. These changes represent the orthogonal shear distortions to which the two-point perspective horopter is subject. Notice, also, that the distortions generate acceptable configurations for the oblique-angled skylight and its optical reflection on the wall and floor. This conformity can be
understood by the fact that the horizontal and vertical projection of the skylight retain their rectangularity, and therefore the skylight, which is the intersection of these two projections, must also do so. The oblique reflection of the skylight, on the other hand, does not retain the same angle around the circle of viewing locations. Nevertheless, this distortion is not noticeable because it remains in accord with the architectural configuration that generates it.

Fig. 6. Reproduction of ‘Fisherman’ by Homero Aguilar (reproduced from www.artfinders.com).
A. Original painting from centered viewpoint. B. View from left side. C View from right side.
6. GENERALIZATION OF THE PERSPECTIVE HOROPTER CONCEPT

Having established the existence of the perspective horopter for rectangularity in the two-point case, one can ask whether this concept generalizes to other scene types. It is obvious that scenes with multiple vanishing points in addition to the first two orthogonal ones will violate the angular preservation concept because the additional angles will be distorted by the orthogonal shear. Hence, there is no angle-preserving horopter of viewing locations for any scene unless all edges are in orthogonal sets. However, the three canonical perspective types do have orthogonal sets of edges. In the case of one-point perspective, Fig. 3 makes it clear that any lateral displacement from the center of projection will distort the angles from rectangularity. However, there is a line along which the viewing location can vary without perturbing the rectangularity of the reconstruction – the line of sight. For any location along the line of sight, a scene geometry can be found for which all the angles remain as right angles, although, as in the two-point case, the reconstructions are subject to orthogonal shear of the squares to rectangles. Indeed, it is generally the case that one cannot reconstruct the scene geometry of one-point perspective without the knowledge of the aspect ratio of some object in the scene, such as a square. If all there is no such familiarity constraint, any one-point construction is subject to an infinite continuum of plausible rectangular reconstructions.

The form of the horopter for the two-point case has already been covered (see Fig. 5).

![Diagram of three-point perspective for a cube](image)

**Fig. 7.** Three-point perspective for a cube for which none of the edges are parallel to the picture plane. Because the three planes are in different spatial orientations 90° apart, three separate vanishing points are required. For correct perspective, the picture should be viewed so that each pair of vanishing points forms 90° at the viewer’s eye (as in Fig. 3).

The most general case of the projection of the cube is triaxial (or three-point) perspective (Fig. 7). Here each edge projects to one of three defined vanishing points in the picture plane. Because the corners all have angles of 90° in space, the vanishing points are each at a 90° angle of view to the viewer. The three vanishing points therefore define the viewing location for the cube. In the example in Fig. 7, the cube is eccentric to the center of view, which is defined by the intersection of three spheres with diameters defined by each pair of vanishing points.

The existence of angle-preserving horopter for the cases of one- and two-point perspective leads to the question of whether there would be a horopter for the three-point construction as well. Because the lines in the scene project to three vanishing points in a triangular configuration, it becomes evident that there is no direction in which the eye can move relative to the original and still retain the three sets of right angles. There is only one location for which the three pairs of vanishing points can all simultaneously subtend 90° at the eye. The extra constraint limits the angle-preserving horopter to a single point – the original center of projection. Any deviation from this point will generate distortions from the intended right angles in one or more of the orthogonal directions.
In summary, therefore, the angle preserving horopter is the line of sight for one-point perspective, a semi-circle in front of the picture plane for two point perspective, and the center of projection for three-point perspective.

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